Math 4200
Monday October 26
3.2 Power series and Taylor series for analytic functions. We begin with the last example from Friday, which we did not get to.

Announcements:

Warm-up and summary/intro problem:

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots.$$

converges uniformly for $|z| \leq R$, so converges to an analytic function $\forall\ z$. Then use the term by term differentiation theorem to show that $f'(z) = f(z)$ and use this and $f(0) = 1$ to identify $f(z)$.

Power series

Consider the *power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C}.$$

Theorem 1:
(i) There exists unique $R \in [0, \infty]$ such that the power series above converges absolutely $\forall\ z$ with $|z - z_0| < R$ and diverges for all z with $|z - z_0| > R$. This value of R is called the *radius of convergence* of the power series.

(ii) For $r < R$, the convergence of the power series is uniform absolute convergence $\forall\ z \in \mathrm{D}(z_0, r)$. Thus f is analytic in $\mathrm{D}(z_0\ ; R)$.

proof: Notice that the radius of convergence is uniquely determined by the two conditions it must satisfy. Define $R := sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n|\, r^n < \infty\}$. We'll show that this number R satisfies the two required contitions so it will be the radius of convergence. It is either a non-negative real number or $+\infty$:

Let $r < R$ and apply the Weirestrass M test in $\bar{\mathrm{D}}(z_0, r)$ to deduce uniform absolute convergence of the power series for $f(z)$ in $\mathrm{D}(z_0, r)$. Thus the power series converges in $\mathrm{D}(z_0, R)$ to an analytic function and we have shown (ii), and half of (i).

Then show the second part of (ii), i.e. divergence when $|z - z_0| > R$, by proving and using

Abel's Lemma: If $sup_{n \in \mathbb{N}} \{|a_n| R_1^n\} = M < \infty$ then $\sum_{n=0}^{\infty} |a_n| r^n < \infty \ \forall \ 0 < r < R_1$

Theorem 2 (differentiation and integration of power series) Consider

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad\qquad a_n, z_0 \in \mathbb{C}$$

with radius of convergence $R > 0$. Then $f'(z)$ can be computed via term by term differentiation; and the antiderivatives $F(z)$ of $f(z)$ can be computed by term by term antidifferentiation (plus an additive constant). The power series for f' and F have the same radius of convergence R as does the power series f.

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad\qquad \forall\ z \in \mathrm{D}(z_0, R)$$

$$F(z) = F(z_0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1}\ (z - z_0)^{n+1} \qquad \forall\ z \in \mathrm{D}(z_0, R)$$

proof: Since the power series for $f(z)$ converges in $\mathrm{D}(z_0, R)$, and uniformly absolutely for any closed subdisk (concentric or not, since each closed subdisk is contained in a closed concentric sub-disk), we deduce from Theorem B′ Friday that the term-by-term differentiated power series for $f'(z)$ also converges in $\mathrm{D}(z_0, R)$, and uniformly for any closed subdisk. Thus the radius of convergence for the $f'(z)$ series is at least the radius of convergence for the f series. But using the characterization of radius of convergence from Theorem 1, the radius convergence for the series for $f'(z)$ is at most R, since the moduli of the terms in the f' series are larger than in the f series:

$$sup\{r \geq 0 \mid \sum_{n=1}^{\infty} n|a_n|\ r^n < \infty\} \leq sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n|\ r^n < \infty\} = R.$$

Thus the radii of convergence for f, f' must be the same. Thus also the radius of convergence for F, $F' = f$ must be equal.

QED.

Theorem 3. (Uniqueness of power series representations) If f is given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C} \qquad a_n, z_0 \in \mathbb{C}$$

with positive radius of convergence R then the power series *is* the Taylor series with

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n = 0, 1, 2, ...$$

In particular, a given analytic function whose domain of analyticity includes z_0 can have only one power series representation centered at z_0.

proof: We know from the previous Theorem that we have

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad \forall\, z \in \mathrm{D}(z_0, R)$$

and inductively, for $k \in \mathbb{N}$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n (z - z_0)^{n-k} \qquad \forall\, z \in \mathrm{D}(z_0, R),\ \forall\, k \in \mathbb{N}.$$

evaluating at z_0 only the first term in the series is nonzero, so

$$f^{(k)}(z_0) = k!\, a_k \quad \Rightarrow a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Theorem 4 If f is analytic in $\mathrm{D}(z_0; R_1)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $\mathrm{D}(z_0; R_1)$. Thus the radius of convergence of the Taylor series is at least R_1. And, one can use this to get an upper bound on the radius of convergence: if $\exists\, z_1$ such that f cannot be extended to be analytic at z_1, then the radius of convergence of the Taylor series is at most $|z_1 - z_0|$, since a larger radius of convergence would imply that a possible domain of analyticity contains z_1.

proof after examples....

Examples

1) Find the Taylor series for $f(z) = e^{z^2}$ at $z_0 = 0$, and its radius of convergence.

2) Find the Taylor series for $f(z) = \dfrac{1}{(z-1)^2}$ at $z_0 = 0$, along with its radius of convergence.

3) Find the Taylor series for $f(z) = \log(1+z)$ at $z_0 = 0$, along with its radius of convergence.

4) Find the Taylor series of $f(z) = \dfrac{1}{z^2 - z - 6} = \dfrac{1}{5}\left(\dfrac{1}{z-3} - \dfrac{1}{z+2}\right)$ at $z_0 = 0$, along with its radius of convergence.

5) Define $\log(z) = \ln|z| + i\,arg(z)$ on the branch domain $0 < arg(z) < 2\pi$. Find the Taylor series for $\log(z)$ at $z_0 = 1 + i$, and find the radius of convergence using the ratio test for absolute convergence. Explain why your answer may seem surprising at first.

Theorem 4 If f is analytic in $D(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges to f in $D(z_0; R)$. Thus the radius of convergence of the power series is at least R.

proof: Let $|z - z_0| \leq r < R_1 < R$, $\gamma(t) = z_0 + R_1 e^{it}$, $0 \leq t \leq 2\pi$, the circle $\zeta - z_0 = R_1$.

Then the Cauchy integral formula reads

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We use geometric series magic:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \dfrac{(z - z_0)}{(\zeta - z_0)}}\, d\zeta$$

using the geometric series for $\dfrac{1}{1 - w}$ with $|w| \leq \dfrac{r}{R_1}$:

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} f(\zeta)\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Because $|f|$ is bounded on γ and

$$\frac{|z - z_0|^n}{|\zeta - z_0|^{n+1}} \leq \frac{1}{R_1}\left(\frac{r}{R_1}\right)^n,$$

the series which is the integrand converges uniformly on γ so we may interchange the summation with the integration, (and then pull each $(z - z_0)^n$ through the integral:

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \int_\gamma \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \quad \frac{1}{2 \pi i} \int_{\gamma} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by the Cauchy integral formula for derivatives!

Q.E.D.